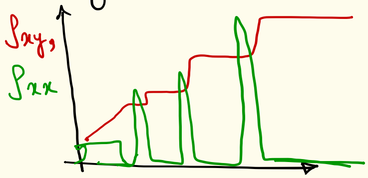
Quantum Hall & Chern-Simons theory

Fractional quantum Hall systems have the following salient features:

(i) They exhibit fractional quantum Hall i.e. $\sigma_{xy} = \dfrac{\nu e^2}{h} = \dfrac{\nu e^2}{2\pi}$.



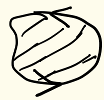
When $\rho_{xy} \neq 0$, $\rho_{xx} = 0 \Rightarrow \sigma_{xx} = 0$ and $\sigma_{xy} = -\dfrac{1}{\rho_{xy}}$.

$\rho_{xx} = 0$ $\Downarrow$ No dissipation

$\sigma_{xx} = 0$ $\Downarrow$ No current along $\vec{E}$.

(ii) They are incompressible : the density is independent of chemical potential.

(iii) They support bulk excitations that have (a) fractional charge (b) fractional statistics (anyons).

(iv) They support edge states that are chiral  .

Remarkably, these properties imply that the low energy description of FQH states is given by Chern-Simons theory of the form $\sim \dfrac{k}{4\pi}\int a \wedge da = \dfrac{k}{4\pi}\int \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$.
$\equiv k/4\pi \; a \wedge da$ where $k \in \mathbb{Z}$.

Such a term is 'topological' because it doesn't depend on the metric (there is no $g_{\mu\nu}$ dependence even on curved manifolds). The action encodes non-trivial commutation relations of non-contractible loops, similar to $Z_1 X_2 = -X_2 Z_1$ and $Z_2 X_1 = -X_1 Z_2$ in toric code. In fact, toric code can also be described by a Chern-Simons theory!

Why Chern-Simons ?

One way to proceed is to follow Zee's 'What else it could be ?' approach. The system has a conserved electric current $\Rightarrow J_\mu = \frac{\varepsilon_{\mu\nu\lambda}}{2\pi} \partial_\nu a_\lambda$. Since time-reversal is explicity broken by the magnetic field and the system does not have an emergent photon, a_μ must be gapped. The only term allowed by these constraints is the C-S term $\sim \frac{k}{4\pi}\int a \wedge da$. As we will soon see, this provides a guage invariant mass $\propto k$ to a. If so, the low energy theory will be

$$\mathcal{L} = \frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + e \underset{\text{E.M. field}}{A_\mu} \underbrace{\frac{\varepsilon_{\mu\nu\lambda} \partial_\nu a_\lambda}{2\pi}}_{J_\mu}$$

$$= \frac{k}{4\pi} a \wedge da + \frac{A \wedge da}{2\pi}$$

E.O.M. for $a \Rightarrow \frac{2k}{4\pi} da + \frac{e\,dA}{2\pi} = 0 \Rightarrow da = -\frac{e\,dA}{k}$

$$\Rightarrow J = \frac{\delta \mathcal{L}}{\partial A} = e\frac{da}{2\pi} = -\frac{e^2}{2\pi k} dA.$$

i.e $$\boxed{J_\mu = -\frac{e^2}{2\pi k} \varepsilon_{\mu\nu\lambda} \partial_\nu A_\lambda}$$

$\Rightarrow J_0 = \text{density} = -\frac{e^2}{2\pi k} B \Rightarrow$ incompressible

$J_x = +\frac{e^2}{2\pi k} E_y \Rightarrow \boxed{\sigma_{xy} = \frac{e^2}{2\pi k}} \Rightarrow$ FQH effect. $(\nu = 1/k)$

Why is the guage field 'a' gapped? Irrelevance of Maxwell term.

The CS term $\frac{a \wedge da}{4\pi}$ has one less derivative than the Maxwell term $(da)^2$, therefore it is more relevant at longest distances. In fact, it provides a mass to the photon.

$$\mathcal{L} = -\frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} + \frac{k}{4\pi} a_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda}$$

$$\text{E.O.M.} \Rightarrow \quad \partial_\mu f^{\mu\nu} = \frac{k}{4\pi} e^2 \varepsilon^{\nu\rho\lambda} f_{\rho\lambda}$$

Defining the dual field strength $\tilde{f}_\mu = \varepsilon_{\mu\nu\lambda} f^{\nu\lambda}$

$$\Rightarrow \quad \left[\partial_\mu \partial^\mu + \left(\frac{ke^2}{2\pi}\right)^2\right] \tilde{f}_\nu = 0$$

i.e. $(\Box + m^2)\tilde{f} = 0$ where $m = \frac{ke^2}{2\pi}$

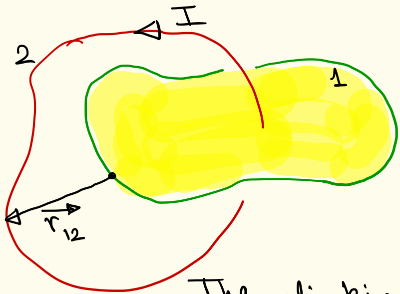
$\Rightarrow$ $\tilde{f}$ has a mass-gap of $\frac{ke^2}{2\pi}$.

Chern-Simons $\Rightarrow$ Anyonic statistics

We ask: if the excitations in 2+1-d have abelian anyonic statistics, what should be there low-energy effective description?

Anyonic statistics imply that when world-lines of particles inter-link, then one obtains a non-zero phase in the path-integral, i.e., a term of the form $e^{2i\theta Lk(C_1, C_2)}$

($\theta = \pi$ for fermions)



Linking number in terms of trajectories

The linking number for two curves is given by:

$$Lk(C_1, C_2) = \frac{1}{4\pi}\int_{C_1}\int_{C_2} \frac{(\vec{r}_{12} \times d\vec{r}_2)\cdot d\vec{r}_1}{r_{12}^3}$$

<u>Physics proof:</u> Let's pass a current $I = 1$ through C_2. Since $\vec{\nabla}\times\vec{B} = \vec{J}$, $\int_{\text{Yellow region}}[\vec{\nabla}\times\vec{B}]\cdot d\vec{s} = \int_{\text{Yellow region}}\vec{J}\cdot d\vec{s}$ $= Lk(1,2)\, I = Lk(1,2)$. The magnetic field at location $\vec{r}_1$ on C_1 $= \frac{1}{4\pi}\int_{C_2}\frac{1}{r_{12}^3}\,\vec{r}_{12}\times d\vec{r}_2 \Rightarrow \int_{\text{Yellow region}}(\vec{\nabla}\times\vec{B})\cdot d\vec{s}$

$$= \oint_{C_1}\vec{B}\cdot d\vec{r}_1 = \frac{1}{4\pi}\int_{C_1}\int_{C_2}\frac{1}{r_{12}^3}\left\{\vec{r}_{12}\times d\vec{r}_2\right\}\cdot d\vec{r}_1$$

Claim: If the particles were coupled to a C-S guage field, then the path integral corresponding to their world-lines would equal $e^{\frac{2\pi i}{k} Lk(C_1, C_2)}$ i.e. $\theta = \frac{\pi}{k}$ above, where $Lk(C_1, C_2)$ is the linking number of world-lines.

Proof: The Lagrangian is :

$$\mathcal{L} = \frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + j_1^\mu a_\mu + j_2^\mu a_\mu$$

where $j^0_\alpha = \delta(\vec{x} - \vec{x}_\alpha(t))$ with $\alpha = 1, 2$

and $j^i_\alpha = \dot{x}^i_\alpha \, \delta(\vec{x} - \vec{x}_\alpha(t))$ $i = x, y$.

The path-integral becomes :

$$Z = \int Da \, e^{i \int \mathcal{L} \, d^2x \, dt}$$

$$= \int Da \, e^{i \int \frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + i \int j_1^\mu a_\mu + i \int j_2^\mu a_\mu}$$

This is a Gaussian integral $\Rightarrow$

$$Z = e^{i \int d^2x \, d\tau \; j_1^\mu \frac{1}{\frac{k}{2\pi} \partial_\lambda \varepsilon^{\mu\nu\lambda}} j_2^\nu}$$

Denoting $\frac{1}{\frac{k}{2\pi} \partial_\lambda \varepsilon^{\mu\nu\lambda}} j_2^\nu = f_2^\mu$ i.e. $\frac{k}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\lambda f_2^\mu = j_2^\nu$

$$Z = e^{i\int d^2x\, dz \; j_1^{\mu} f_2^{\mu}}$$

f_2 can be solved since it's of the Biot-Savart form $\nabla \times B = J$. $\Rightarrow$

$$f_2^{\mu} = \frac{1}{2k} \int d^3y \, \frac{\varepsilon_{\mu\nu\lambda}\, \partial_\nu j_2^{\lambda}}{|\vec{x}-\vec{y}|}$$

$$\Rightarrow \quad Z = e^{i\int d^3x \, \frac{j_1^{\mu} \, \varepsilon_{\mu\nu\lambda}\, \partial_\nu j_2^{\lambda}}{2k \;\, |\vec{x}-\vec{y}|}}$$

Integrating by parts:

$$Z = e^{-\frac{i}{2k}\int d^3x\, d^3y \, \frac{(x-y)^{\nu} j_1^{\mu}(x)\, \varepsilon_{\mu\nu\lambda}\, j_2^{\lambda}(y)}{|\vec{x}-\vec{y}|^3}}$$

Since j_1, j_2 are non-zero only along the paths C_1, C_2 $\Rightarrow$

$$Lk(C_1, C_2) = \frac{1}{4\pi}\int d^3x\, d^3y \, \frac{(x-y)^{\nu}}{|\vec{x}-\vec{y}|^3} \, j_1^{\mu}(x)\, \varepsilon_{\mu\nu\lambda}\, j_2^{\lambda}(y)$$

$$\boxed{\Rightarrow \quad Z = e^{-\frac{2\pi i}{k} Lk(C_1, C_2)}}$$

Thus, C-S action leads to anyonic exchange statistics $= \frac{\pi}{k}$ for excitations that carry a unit charge of 'a' field.

Chern-Simons action $\Rightarrow$ Fractional charge

Let's include the external E.M. field $\vec{A}$ in the above calculation. The full Lagrangian is:

$$\mathcal{L} = -\frac{k}{4\pi}\, a\,da + e\,\frac{A\,da}{2\pi} + j_\mu a_\mu$$

[We have sent $k \to -k$ so that the anyons statistics is $+LK(C_1, C_2)$ above].

Again integrating out $a \Rightarrow$

$$\mathcal{L}_{eff} = \frac{\pi}{k}\, \tilde{j}_\mu \left[\frac{\varepsilon^{\mu\nu\lambda}}{\partial_\nu}\right] \tilde{j}_\lambda$$

where $\displaystyle \tilde{j}_\mu = j_\mu - \frac{e\, \varepsilon_{\mu\nu\lambda}\, \partial_\nu A_\lambda}{2\pi}$

To calculate charge of the excitation associated with j, we need to isolate the term proportional to $A_\mu j^\mu$. That term is:

$$2 \times \frac{\pi}{k}\, \frac{j_\mu A_\mu}{2\pi} = \frac{e\, j_\mu A_\mu}{k}$$

Since charge at location $x = \dfrac{\delta \mathcal{L}}{\delta A_0} = \dfrac{e\, j_0}{k}$

$\Rightarrow$ The quasiparticle that carries a unit charge of 'a' field, carries $\frac{e}{k}$ electric charge.

$\Rightarrow$ fractional charge excitations.

Where is electron? Statistics and charge of a bundle of excitations.

Consider an excitation that carries l units of charge for the 'a' field. The current j for such an excitation will contribute a term $\int l\, j^{\mu} a_{\mu}$ to the action. $\Rightarrow$ statistics of such a excitation $= \pi \frac{l^2}{k}$. We see that $l = k$. corresponds to statistics $= \pi k$. Similarly, Charge of such an excitation $= \frac{e}{k} \times k = e$.

For this excitation to correspond to an electron, $\pi k \equiv \pi$ i.e. k <u>must</u> be an odd integer. If so, we have identified the electron as a bundle of k elementary excitations.

Relation to microscopics / wavefn

Above we argued that the CS action

$$S = \int \left(-k \frac{a\,da}{4\pi} + \frac{A\,da}{2\pi} + l\, a_\mu j^\mu \right)$$

corresponds to a topologically ordered state where a particle with unit charge of 'a' has exchange statistics $\theta = \frac{\pi}{k}$, and a bound state of k such particles corresponds to the electron. A ground state wave-f^n that is described by this action is the Laughlin state

$$\psi = \prod_{i<j} (z_i - z_j)^k \, e^{-|z|^2/4l_B^2}$$

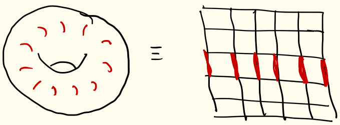
[$l_B^2 = \frac{c\hbar}{eB}$ is the magnetic length].

See Wen's book 7.2.2 and 7.2.3 for details

Loop algebra and Topological degeneracy on Torus

The abelian CS theory $\int \frac{k\, a da}{4\pi}$ resembles a free theory and the E.O.M. is $da = 0$ which suggests that the theory is trivial with no interesting correlation fns. This is clearly incorrect as discussed above — the CS term leads to anyons statistics. More formally, it the Correlation fns of wilson loops $e^{i\int_{c_1} \vec{a}\cdot d\vec{l}}\, e^{i\int_{c_2} \vec{a}\cdot d\vec{l}}$ are non-trivial and equal $e^{-\frac{2 i \pi}{k} Lk(c_1, c_2)}$, as discussed earlier.

The situation is identical to the ground state of $\mathbb{Z}_2$ guage theory. There, we saw that in the ground state, flux $= 0$ through every plaquette. This still allowed for four non guage-equivalent configurations such:



. In terms of operators, the non-trivial loop configs. corresponded to the operators Z_1, X_1, Z_2, X_2 which had non-trivial commutation relations leading to topological degeneracy on torus.

Lets consider such an approach for the CS theory. The main-difference is that in the toric code / $\mathbb{Z}_2$ guage theory the commutation relations were

$[A_x, E_x] = i$, $[A_y, E_y] = i$ and $[A_x, A_y] = [E_x, E_y] = 0$

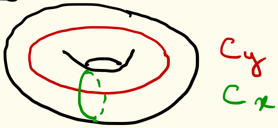
where A, E are on the same link.

In contrast, the CS action is $\sim \frac{2\pi i}{k} a_x \partial_t a_y$ (choosing $a_0 = 0$ guage) $\Rightarrow$ $[\hat{a}_x(r), \hat{a}_y(r')] = \frac{2\pi i}{k}\delta^2(r-r')$, thus the field d.o.f are "halved" compared to a conventional guage theory.

Let's define loop operators

$$\hat{W}_i = e^{i\int_{C_i} \hat{a}\cdot dl}$$

where C_x winds around y-direction (and thus $W_x \sim$ exp(i flux through the non-contractible x-hole). Similarly C_y winds along x-direction.



Using the commution relations for $\hat{a}$,

$$\hat{W}_x \hat{W}_y = e^{\frac{2\pi i}{k}} \hat{W}_y \hat{W}_x .$$

(recall BCH formula for operators whose commutator is a c-number $A\,B = e^{A+B+\frac{1}{2}[A,B]}$).

The above non-commutivity of W_x, W_y imply that ground state is k-fold degenerate on a torus. To see this consider a g.s. labeled $|0\rangle$ with $\hat{W}_x$ eigenvalue 1.

$$\hat{W}_x |0\rangle = |0\rangle$$

Consider the state $\hat{W}_y|0\rangle$. The Hamiltonian of CS theory is zero, so it clearly is still an eigenstate, the question is whether it's a new one?

$$\hat{W}_x \hat{W}_y |0\rangle = \hat{W}_y \hat{W}_x e^{2\pi i/k} |0\rangle$$

$$= \hat{W}_y e^{2\pi i/k} |0\rangle.$$

Thus, $\hat{W}_y|0\rangle$ is again an eigenvector of $\hat{W}_x$ but with a different eigenvalue, namely, $e^{2\pi i/k}$. We can denote $\hat{W}_y|0\rangle \equiv |1\rangle$. Repeting the process $k-1$ times, we obtain k orthogonal ground states $|0\rangle, |1\rangle, \ldots |k-1\rangle$.

Quantization of C-S theory by mapping to QM

Since $da=0$ locally, the nontrivial dependence of $\vec{a}$ is only on time modulo the fact that the physical degrees of freedom are $e^{i\oint \vec{a}\cdot d\vec{l}}$ and not $\vec{a}\cdot d\vec{l}$.

Writing $a_x = \dfrac{\theta_x(t)}{L_x}$, $a_y = \dfrac{\theta_y(t)}{L_y}$, the action becomes

$$S = \frac{1}{L_x L_y} \int d^2x \, dt \, \frac{2k}{4\pi} \theta_x \dot{\theta}_y = \frac{k}{2\pi} \int \theta_x \dot{\theta}_y \, dt$$

with the identification $\theta_x \equiv \theta_x + 2\pi$, $\theta_y \equiv \theta_y + 2\pi$

Since $e^{i\oint_{C_1} a \cdot dl}$, $e^{i\oint_{C_2} \vec{a} \cdot d\vec{l}}$ are unchanged under such a transformation. Compare the action with that of particle in a uniform magnetic field $B\hat{z}$ in 2d. $\vec{A} = \frac{B}{2}(y, -x) \Rightarrow S = \int \vec{A} \cdot \vec{v}$

$= \frac{B}{2}\int (y\dot{x} - x\dot{y})\, dt = -B\int x\dot{y}\, dt$. Thus, our action corresponds to a magnetic field of $B = \frac{k}{2\pi}$. Recall that particle in a magnetic field has degenerate spectrum with degeneracy $= \frac{\text{Total flux}}{2\pi}$. In our problem, flux $= B \times \int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y$

$= \frac{k}{2\pi} \times 4\pi^2 = 2\pi k$

$$\Rightarrow \boxed{\text{Degeneracy} = \frac{2\pi k}{2\pi} = k}$$

Wave-functions :

Since the Hamiltonian is zero, one might think that any function is an eigenfunction. However, the function needs to satisfy $\psi(\theta_x, \theta_y)$

$= \psi(\theta_x + 2n\pi, \theta_y + 2m\pi)$ where $m, n \in \mathbb{Z}$.

Since θ_x, θ_y don't commute, the eigenfn can only be a fn of one of them. Let's work with $\psi(\theta_x)$

write $\psi(\theta_x) = \sum_n a_n e^{in\theta_x}$ which automatically satisfies $\psi(\theta_x) = \psi(\theta_x + 2\pi)$. Fourier transforming,

$$\psi(p) = \sum_n a_n \delta(p-n)$$

where p is canonically conjugate to θ_x. Of course, $p = \frac{k\theta_y}{2\pi}$.

$$\Rightarrow \psi(\theta_y) = \sum_n a_n \delta\left(\frac{k\theta_y}{2\pi} - n\right)$$

The requirement $\psi(\theta_y) = \psi(\theta_y + 2\pi) \Rightarrow$

$$a_n = a_{n+k}.$$

Therefore, there are just k independent a_n's corresponding to k different topologically degenerate sectors.

Toric code as a Chern-Simons Theory

Above we saw that the CS theory $\frac{k}{4\pi}\int a\, da$ leads to $W_x W_y = e^{\frac{2\pi i}{k}} W_y W_x$. In toric code, the algebra of loop operator was $X_1 Z_2 = -Z_2 X_1$ and $X_2 Z_1 = -Z_1 X_2$. This almost looks like CS theory at $k=2$ except there are twice as many loop operators. Therefore consider a CS theory with two guage fields a_1, a_2:

$$S = \frac{2}{4\pi}\int a_1\, da_2$$

following the same procedure as above, the operators $W_{1x} = e^{\int_{c_x} i\, \vec{a}_1 \cdot d\vec{\ell}}$, $W_{2x} = e^{i\int_{c_x} \vec{a}_2 \cdot d\vec{\ell}}$, $W_{1y} = e^{i\int_{c_y} \vec{a}_1 \cdot d\vec{\ell}}$, $W_{2y} = e^{i\int_{c_y} \vec{a}_2 \cdot d\vec{\ell}}$ satisfy the following relations:

$$\boxed{\begin{aligned} W_{1x} W_{2y} &= -W_{2y} W_{1x} \\ W_{2x} W_{1y} &= -W_{1y} W_{2x} \end{aligned}}$$

Thus, one may identify $W_{1x} = X_1$, $W_{2y} = Z_2$, $W_{2x} = X_2$, $W_{1y} = Z_1$ from the toric code.

Thus, the operator $e^{i\int_A^B \vec{a}_1 \cdot d\vec{l}}$ creates a $\mathbb{Z}_2$ magnetic flux (m) at A and B, and $e^{i\int_A^B \vec{a}_2 \cdot d\vec{l}}$ creates a $\mathbb{Z}_2$ electric charge (e) at A and B.

Similarly, the mutual statistics between e and m particles is a result of the fact that

$$\left\langle e^{i\oint_{C_1} a_1 \cdot dl} \; e^{i\oint_{C_2} \vec{a}_2 \cdot d\vec{l}} \right\rangle$$

$$= e^{i\pi \, Lk(C_1, C_2)}$$ where C_1, C_2 are closed curves and thereby correspond to space-time trajectories of an m and an e particle respectively.

More general Chern-Simons theories and their topological degeneracy.

Above, the toric code example had two CS fields a_1, a_2. More generally, one can have:

$$S = -\frac{K_{IJ}}{4\pi}\int a_I \, da_J + \int l_I \, a_{I\mu} \, j^\mu$$

$$+ \frac{e}{2\pi}\int q_I \, A \, da^I .$$

where K is a $n\times n$ matrix with integer coefficients.

A quasiparticle is distinguished by it's charges l_I under guage field a_I, and the electron current is given by $\sum_I q_I \frac{d a_I}{2\pi}$ where q_I is called the (electron) charge vector. One can show that

filling $= \sigma_{xy} = q^T K^{-1} q$.

Charge and statistics of a quasiparticle:

charge $\quad Q_l = -e\, l^T K^{-1} q$.

statistics $\quad \theta_l = \pi\, l^T K^{-1} l$.

The degeneracy on a torus can be obtained in the similar manner as above:

One quantizes the CS theory $-\frac{K_{IJ}}{4\pi}\int a_I\, da_J$ by writing $a_{Ix} = \frac{\theta_x^I}{L_x}$, $a_{Iy} = \frac{\theta_y^I}{L_y}$. This leads to

$$S = \int \frac{K_{IJ}}{2\pi}\, \theta_x^I\, \dot{\theta}_y^J$$

$$\Rightarrow \quad [\theta_x^I, \theta_y^J] = 2\pi i\, (K^{-1})_{IJ}$$

Diagonalizing the K matrix, one obtains

n sets of canonically conjugate variables $\tilde{\theta}^\alpha_x, \hat{\theta}^\alpha_y$ which satisfy $[\tilde{\theta}^\alpha_x, \hat{\theta}^\alpha_y] = \frac{2\pi i}{k_\alpha}$

where k_α is an eigenvalue of K. A set contributes topo. degeneracy k_α. Total degeneracy $= \prod_\alpha k_\alpha = |\det K|$.

Not all different K matrices correspond to a different topological phase of matter. Any transformation of the form $K \to W K W^T$ and $q \to Wq$ where $W \in SL(n, \mathbb{Z})$ (= integer matrices with $\det = 1$) just corresponds to a change of basis.

Moreover, even K matrices with different matrix size can describe same physics due to various dualities. We will see an example later.
